

Properties of the static, spherically symmetric solutions in the Jordan, Brans-Dicke theory.

S.M.KOZYREV

e-mail: Kozyrev@e-centr.ru

ABSTRACT.

We have studied the properties of the static, spherically symmetric solution of Jordan, Brans-Dicke theory. An exact interior solution for standard space-time line element in the Schwarzschild form is obtained.

1. INTRODUCTION.

Scalar-tensor and vector-tensor gravity theories provide the most natural generalizations of General Relativity by introducing additional fields. In these theories of gravity, the field equations are even more complex than in General Relativity. We restrict our discussion to the Jordan-Brans-Dicke (JBD) theory [1, 2] which among of all the alternative theories of classical Einstein's gravity, is the most studied and hence the best known. This theory can be thought of as a minimal extension of general relativity designed to properly accommodate both Mach's principle [3] and Dirac's large number hypothesis [3]. Namely, the theory employs the viewpoint in which the scalar potential is the analog of gravitational permittivity is allowed to vary with space and time which defined using Newton's gravitational constant as $G = 1/\phi$. JBD theory contains a massless scalar field ϕ and a dimensionless constant ω that describes the strength of the coupling between ϕ and the matter. The field equations of the JBD theory are obtained by the similar variational method as the Einstein theory, and are given as following:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi}{\phi}T_{\mu\nu} - \frac{\omega}{\phi^2} \left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\phi_{,\lambda}\phi^{,\lambda} \right) - \frac{1}{\phi} \left(\phi_{;\mu;\nu} - g_{\mu\nu}\phi_{;\lambda}^{;\lambda} \right), \quad (1)$$

$$\phi_{;\lambda}^{;\lambda} = -\frac{8\pi}{3+2\omega}T, \quad (2)$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter and ω is the coupling parameter of the scalar field.

It is usually believed that the post-Newtonian expansions of JBD theory reduces to general relativity when the JBD parameter $|\omega| \rightarrow \infty$ (see e.g. Ref. [4]), thus the field equations of gravitation coincide completely with those of general relativity by replacing ϕ with Newton's gravitational constant $G \equiv 1/\phi$. The JBD field ϕ is believed to exhibit the asymptotic behavior

$$\phi = \phi_0 + \mathcal{O}\left(\frac{1}{\omega}\right), \quad (3)$$

(where ϕ_0 is a constant).

However, the standard assumption the JBD solutions with $T = 0$ generically fail to reduce to the corresponding solutions of general relativity when $\omega \rightarrow \infty$; a number of exact JBD solutions have been reported not to tend to the corresponding general relativity solutions [5], [6], [7]. Moreover, some authors [8], [9], [10] reported that the asymptotic behavior of the JBD field is not (3) when the energy-momentum tensor $T = T^\mu_\mu$ vanishes. In this case $T = 0$, asymptotic behavior of the scalar field becomes

$$\phi = \phi_0 + \mathcal{O}\left(\frac{1}{\sqrt{\omega}}\right). \quad (4)$$

These situation are alarming since the standard belief that JBD theory always reduces to general relativity in the large ω limit is the basis for setting lower limits on the ω -parameter using celestial mechanics experiments [11]. To make the situation worse, the Hawking theorem [12] states that the Schwarzschild metric is the only spherically symmetric solution of the vacuum JBD field equations. The proof of this theorem goes through the fact that the JBD scalar field ϕ must be constant outside the black hole and the use of the weak energy condition.

The progress in the understanding of scalar-tensor theories of gravity is closely connected with finding and investigation of exact solutions. Shortly after JBD theory was proposed, Heckmann obtained parametric form of the exact static vacuum solution to the JBD equations [1]. Later Brans [2] find the static, spherically symmetric, vacuum solution of the JBD equations in harmonic coordinates. However, not much study has been done to give physical interpretations to the constant parameters appearing in the Heckmann and Brans solutions [13].

As an example, one can consider the static, spherically symmetric, vacuum Brans solution [2] given by

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2) \right], \quad (5)$$

$$e^{2\alpha} = \left(\frac{1 - B/r}{1 + B/r} \right)^{2/\sigma}, \quad (6)$$

$$e^{2\beta} = \left(1 + \frac{B}{r}\right)^4 \left(\frac{1 - B/r}{1 + B/r}\right)^{2(\sigma - C - 1)/\sigma}, \quad (7)$$

$$\phi = \phi_0 \left(\frac{1 - B/r}{1 + B/r}\right)^{-C/\sigma}, \quad (8)$$

where

$$\sigma = \left[(C + 1)^2 - C \left(1 - \frac{\omega C}{2}\right)\right]^{1/2}, \quad (9)$$

$$B = \frac{M}{2C^2\phi_0} \left(\frac{2\omega + 4}{2\omega + 3}\right)^{1/2}, C = -\frac{1}{2\omega}, \quad (10)$$

and where M is the mass. This solution reduces to the Schwarzschild solution of Einstein's theory for $\omega \rightarrow \infty$ [14]. However, choices of the constant C different from the one in Eq. (10) are possible, and for arbitrary values of the parameter C the solution (5)-(10) does not reduce to the Schwarzschild solution when $\omega \rightarrow \infty$.

The values of the parameters M , C , ω in the Brans solution are not arbitrary; it was shown that the positivity of the tensor mass puts bounds on C and σ [13]. However, complete understanding of the relationships between the parameters M , C and ω , and their respective ranges of admissible values is not yet available.

The purpose of this paper is to present a method for finding exact solutions to the JBD field equations for the static, spherically symmetric case. Throughout the paper, we use the metric signature $- + + +$, and Latin (Greek) indices take values $1\dots 3$ ($0\dots 3$); the Riemann tensor is given in terms of the Christoffel symbols by $R_{\mu\nu\rho}^{\sigma} = \Gamma_{\mu\rho,\nu}^{\sigma} - \Gamma_{\nu\rho,\mu}^{\sigma} + \Gamma_{\mu\rho}^{\alpha} \Gamma_{\alpha\nu}^{\sigma} - \Gamma_{\nu\rho}^{\alpha} \Gamma_{\alpha\mu}^{\sigma}$, the Ricci tensor is $R_{\mu\rho} \equiv R_{\mu\nu\rho}^{\nu}$, and $R = g^{\alpha\beta} R_{\alpha\beta}$. We use units in which the speed of light and Newton's constant assume the value unity.

2. METHOD OF SOLUTION (COORDINATE AND VARIABLE CHOICES).

We assume that spacetime is static; the metric and the scalar field can be chosen such that

$$g_{\mu\nu,0} = \frac{\partial g_{\mu\nu}}{\partial t} = 0; \phi_{,0} = 0; g_{0i} = 0.$$

As we have already mentioned we consider standard static and spherically symmetric space-time with a line element in the form

$$ds^2 = -e^\psi dt^2 + e^\lambda dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2). \quad (11)$$

For the further simplification of the problem of solving the field equations (1), (2) we will replace variable r by $r(\psi)$, then the line element (11) take a form

$$ds^2 = -e^\psi dt^2 + e^{\lambda(\psi)} r'(\psi)^2 d\psi^2 + r(\psi)^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (12)$$

and we find from equations (1), (2) that

$$\begin{aligned} R_{11} = & \frac{1}{4} \left(\left(1 + \frac{4r'}{r} \right) \lambda' \right) - 1 + \left(\frac{1}{r'} - \frac{4}{r} \right) r'' = \\ & \frac{1}{2\phi^2} \left(2\omega\phi'^2 + \phi \left(2\phi'' - \frac{\phi' (r'' + r'\lambda')}{r'} \right) \right), \end{aligned} \quad (13)$$

$$R_{22} = R_{33} = 1 + \frac{e^{-\lambda}}{2r'} \left(r (r' (\lambda' - 1) - r'') - 2r'^2 \right) = \frac{r\phi'}{\phi} e^{-\lambda}, \quad (14)$$

$$R_{00} = \frac{e^{\psi-\lambda}}{4r r'^2} \left(4r'^2 - r (r' (\lambda' - 1) + r'') \right) = -\frac{e^{\psi-\lambda}\phi'}{2\phi r'}, \quad (15)$$

$$\phi_{;\lambda}^{\lambda} = \frac{4\phi'}{r} + \frac{\phi'}{r'} - \frac{\lambda'\phi'}{r'} - \frac{\phi'r''}{r'^2} + \frac{2\phi''}{r'} = 0. \quad (16)$$

where ψ is a new variable and the primes denote derivatives with respect to ψ . Making use of equation (16) in (15) they simplify to

$$\frac{\phi'}{2\phi} - \frac{\phi''}{2\phi'} = 0.$$

Thus we obtain

$$\phi = \text{const } e^{a\psi}, \quad (17)$$

where a is a arbitrary constant. Using the asymptotic condition in infinity we have $\text{const} = 1$. In the case $a = 0$ one can find solution of equations (13) - (16)

$$r = \frac{b}{e^\psi - 1}, \quad e^\lambda = e^{-\psi}, \quad \phi = 1, \quad (18)$$

that identical with the Schwarzschild solution of the Einstein theory.

When $a \neq 0$ making use equations (13), (15) and (16) we eliminate λ' , ϕ'' and obtain for r

$$r = c e^{-\frac{(1+2a)(\psi+2b)}{2}} \sec \left(\frac{(\psi + 2b) k}{2} \right), \quad (19)$$

where b and c arbitrary constants, and $k = \sqrt{-1 - 2a(1 + a(2 + \omega))}$. Finally from equation (14) we have

$$e^\lambda = \frac{k^2}{a - a^2\omega + (1 + a(3 + a(4 + \omega))) \cos \gamma - k(1 + 2a) \sin \gamma}, \quad (20)$$

where $\gamma = (k(2b + \psi))$.

3. ASYMPTOTIC BEHAVIOR.

We will study now the geometrical properties of the metric (19), (20) for given values of the arbitrary parameters a , b and c . Obviously, the metric (19) and (20) must be asymptotically flat. It is enough to show that the metric components behave in an appropriate way at large r -coordinate values, e.g. $g_{\mu\nu} = \eta_{\mu\nu} + O(1/r)$ as $r \rightarrow \infty$. In this case we have

$$\lim_{\psi \rightarrow 0} r(\psi) = \infty; \quad \lim_{\psi \rightarrow 0} \lambda(\psi) = 0$$

then we obtain from (19), (20)

$$c e^{-b(1+2a)} \sec \left(\frac{2bk}{2} \right) = \infty, \quad (21)$$

$$\frac{k^2}{a - a^2\omega + (1 + a(3 + a(4 + \omega))) \cos(2bk) - k(1 + 2a) \sin(2bk)} = 1. \quad (22)$$

One can see there is not solution of the system equations (21), (22). In this case only the Schwarzschild metric (18) is the spherically symmetric solution of vacuum JBD field equations and the JBD scalar field ϕ must be constant outside the matter distributions. However, for the internal solutions increases the interaction between scalar, and tensor fields describing gravitation.

4. INTERIOR SOLUTIONS

Solving of scalar-tensor theory equations in the presence of a matter is a difficult task due to their complexity in the general case. In the simple case of a perfect-fluid spherically symmetric model with equation of state $p = \varepsilon\rho$ much progress has been achieved in finding exact solutions [15], [16]. When the energy-momentum tensor is specialized to that of a perfect fluid and equation of state chosen for very high density we derive the relation between ϕ and g_{00} . In this case we can express the R_0^0 component of equations (1) in the form:

$$-\frac{1}{2} \left[\sqrt{-g} \phi (\ln g_{00})^{,k} \right]_{,k} = 8\pi \left(T_0^0 - \frac{\omega + 1}{2\omega + 3} T \right) \sqrt{-g}. \quad (23)$$

Furthermore in the static case equation (2) simplifies to

$$\left[\sqrt{-g} \phi (\ln \phi)^{,k} \right]_{,k} = \frac{8\pi T \sqrt{-g}}{2\omega + 3}. \quad (24)$$

Combining equations (23), (24) and making use of equation of state $p = \varepsilon\rho$, we obtain

$$\left[\sqrt{-g} \phi \left(\ln \left(\frac{\phi}{g_{00}^q} \right) \right)^{,k} \right]_{,k} = 0, \quad (25)$$

where

$$q = \frac{3\varepsilon - 1}{(2\omega + 3) + (\omega + 1)(3\varepsilon - 1)}.$$

Equation (25) is also valid outside matter with q an arbitrary constant. If ϕ/g_{00} tends uniformly to a limit at infinity and its second derivatives exist everywhere, then the solutions to equation (25) is [16]

$$\phi = e^{a\psi}, \quad (26)$$

where a arbitrary constant. This standard tenet about the relation between ϕ and g_{00} can be false, there are a number of exact JBD vacuum solution with a $\phi = \text{const}$. For example, in conformity with Hawking theorem [12] only the Schwarzschild metric is the spherically symmetric solution of the vacuum JBD field equations. At the same time inside the matter arise the coupling between the scalar field with gravity. Then the relation between ϕ and g_{00} in general case is more complexity then (26). For example, in a special case when equation of state is $p = \varepsilon\rho$, $\omega = -(1 + 6\varepsilon) / (1 - 3\varepsilon)$ and $g_{rr}=\text{const}$ one can find relation between ϕ and g_{00} in the form:

$$g_{00} = \phi^{-2} \left(-\frac{C_1}{r} + C_2 \right)^2, \quad (27)$$

where C_1 and C_2 is a arbitrary constant.

However, in the present work, we investigate only $\phi = e^{a\psi}$ solutions to equation (25). In the paper [15], Bruckman and Kazes demonstrated that, in the case of cold ultrahigh-density static configurations with relation (26) between ϕ and g_{00} , solutions to the Brans-Dicke equations have the metric component g_{rr} is a constant. Assuming that space-time is static, spherically symmetric, and the metric is chosen in the form (11), the relation between ϕ and g_{00} is $\phi = e^{a\psi}$ and $g_{rr}=1$, we find static solution of equations (1), (2):

$$\psi = c_2 + \frac{2(1+2a) \log(r^2(1+4a^2(1+\omega)) + 4(1+2a)c_1)}{1+4a^2(1+\omega)}. \quad (28)$$

Moreover using the relation (26) we can find solution of equations (13)-(16) with $\rho = 0$ and $p \neq 0$

$$e^\psi = \left(-\frac{4\pi r^2(6+\omega)}{3+2\omega} \right)^{\frac{2(1+\omega)}{2+\omega}},$$

$$e^\lambda = \frac{6+\omega}{2+\omega},$$

$$p = e^{-\frac{\psi}{2}},$$

$$\phi = \left(-\frac{4\pi r^2(6+\omega)}{3+2\omega} \right)^{\frac{1}{2+\omega}}.$$

There are not analogs of this solution in Einstein theory.

5. DISCUSSIONS

Analyzing the static, spherically symmetric solution of Jordan, Brance - Dicke theory we find the following. The Schwarzschild metric is the only spherically symmetric solution of the vacuum JBD field equations. Moreover, when the energy-momentum tensor $T=T^\mu_\mu$ vanishes one can use $\phi = \text{const}$ scalar field outside the matter and the field equations of the JBD theory become the similar as the equations of the Einstein theory. Then in empty space there is not a difference between scalar-tensor theories (as well as vector-metric theories [17]) and Einstein theory. In this case in empty space celestial-mechanical experiments to reveal a difference between scalar-tensor theories and Einstein theory is not presented possible. However, scalar field inside the matter has characteristics like gravitation permittivity of material similar electromagnetic permittivity of material in Maxwell theories of electromagnetism [18]. So, in the case of our suggestion is correct, effects considered like the fifth force and the differences between JBD and an Einstein's theories should be experimentally tested in substance.

ACKNOWLEDGMENTS

This article is dedicated to the memory of Dr. V.I.Bashkov. He was of great help teaching me some of the interesting of scalar gravity and discussing various alternative ideas and its role in the development of gravity theories.

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